THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 University Mathematics 2017-2018
Suggested Solution to Assignment 5

1. $f(x) = \dfrac{|x|(x+16)}{x-2}$ for $x \neq 2$ is

$$f(x) = \begin{cases} \dfrac{x(x+16)}{x-2}, & \text{if } x \geq 0, x \neq 2. \\ \\ -\dfrac{x(x+16)}{x-2}, & \text{if } x < 0. \end{cases}$$

(a) (i)

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x+16}{x-2} = -8,$$
$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} -\frac{x+16}{x-2} = 8.$$

The two are NOT equal. Therefore $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist, i.e. $f(x)$ is not differentiable at $x = 0$.

(ii)

$$f'(x) = \begin{cases} \dfrac{(x-8)(x+4)}{(x-2)^2}, & \text{if } x > 0, x \neq 2. \\ \\ -\dfrac{(x-8)(x+4)}{(x-2)^2}, & \text{if } x < 0. \end{cases}$$

$$f''(x) = \begin{cases} \dfrac{72}{(x-2)^3}, & \text{if } x > 0, x \neq 2. \\ \\ -\dfrac{72}{(x-2)^3}, & \text{if } x < 0. \end{cases}$$

(b) From (a) we have

(i) $f'(x) > 0$ for $x > 8$ or $-4 < x < 0$; $f'(x) < 0$ for $x < -4$, $0 < x < 2$ or $2 < x < 8$.

(ii) $f''(x) > 0$ for $x < 0$ or $x > 2$; $f''(x) < 0$ for $0 < x < 2$.

(c) Candidates for local extremal points are $x = -4$, $x = 0$ and $x = 8$. It is not hard to see $x = -4$ and $x = 8$ are relative minimum points of $f(x)$, corresponding to points $(-4, -8)$, $(8, 32)$ on the graph; $x = 0$ is a relative maximum point of $f(x)$, corresponding to the point $(0, 0)$ on the graph. There are no inflection points for $f(x)$, since $f''(x) = 0$ has no solution.

Notice. Although $f'(x)$ changes signs passing through $x = 0$, but $f(x)$ is not differentiable at $x = 0$, therefore $x = 0$ does not count as an inflection point.

(d) $\lim\limits_{x\to 2} f(x) = \infty$, therefore $x = 2$ is a vertical asymptote. When $x >> 0$,

$$f(x) - (x+18) = \frac{36}{x-2} \to 0 \quad \text{as} \quad x \to +\infty.$$

When $x << 0$,

$$f(x) - (-x - 18) = -\frac{36}{x-2} \to 0 \quad \text{as} \quad x \to -\infty.$$

Therefor $y = x + 18$ and $y = -x - 18$ are oblique asymptotoes.

(e) See Figure **??**.

2. (a) The limit is of type $\frac{0}{0}$, and satisfies the conditions of « L'Hôpital's rule », hence

$$\lim_{x\to 0} \frac{e^{2x} - 1 - 2x}{x} = \lim_{x\to 0} \frac{(e^{2x} - 1 - 2x)'}{x'} = \lim_{x\to 0} \frac{e^{x} - 2}{1} = -1.$$

(b) The limit is of type $\frac{0}{0}$, and satisfies the conditions of « L'Hôpital's rule », hence

$$\lim_{x\to 0} \frac{\tan^{-1} x}{2x} = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{2} = \frac{1}{2}.$$

(c) $\lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}}$ is of type $\frac{\infty}{\infty}$, and satisfies the conditions of « L'Hôpital's rule », hence

$$\lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{2(\ln x)\cdot \frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-2\ln x}{\frac{1}{x}}.$$

This is of type $\frac{\infty}{\infty}$, and satisfies the conditions of « L'Hôpital's rule », hence

$$\lim_{x\to 0^+} \frac{-2\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{-2}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} 2x = 0.$$

(d) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{x}{x-1} \right) = \frac{x - 1 - x\ln x}{(x-1)\ln x}$ is of type $\frac{0}{0}$, and satisfies the conditions of « L'Hôpital's rule », hence

$$\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{x}{x-1} \right) = \lim_{x\to 1} \frac{-\ln x - 1 + 1}{\ln x + (x-1)\frac{1}{x}} = \lim_{x\to 1} \frac{-\ln x}{\ln x - \frac{1}{x} + 1}.$$

This is of type $\frac{0}{0}$, and satisfies the conditions of « L'Hôpital's rule », hence

$$\lim_{x\to 1} \frac{-\ln x}{\ln x - \frac{1}{x} + 1} = \lim_{x\to 1} \frac{-\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{-1}{1+1} = \frac{-1}{2}.$$

(e) The limit does not exist.

Let $f(x) = \left(\frac{\sin^2 x}{x} \right)^{\frac{1}{x^2}}, g(x) = \left(\frac{1}{x} \right)^{\frac{1}{x^2}}$.

When $x = (n + \frac{1}{2})\pi, \sin^2 x = 1$. So $f((n + \frac{1}{2})\pi) = g((n + \frac{1}{2})\pi)$ for all positive integer n.

Let $y = \left(\frac{1}{x} \right)^{\frac{1}{x^2}}$, then $\ln y = \frac{1}{x^2} \ln \frac{1}{x}$. According to « L'Hôpital's rule », we have

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln \frac{1}{x}}{x^2} = \lim_{x\to\infty} \frac{-\frac{1}{x}}{2x} = 0$$

So

$$\lim_{x\to\infty} \left(\frac{1}{x} \right)^{\frac{1}{x^2}} = e^0 = 1$$

Therefore the sequence $f((n + \frac{1}{2})\pi)$ converges to 1. However the sequence $f(n\pi)$ converges to 0. (Since all terms are 0) Therefore we conclude that the limit does not exist.

(f) Let $y = x^{\frac{1}{1-x}}$, then $\ln y = \dfrac{1}{1-x}\ln x$. According to « L'Hôpital's rule », we have

$$\lim_{x\to 1}\ln y = \lim_{x\to 1}\frac{\ln x}{1-x} = \lim_{x\to 1}\frac{\frac{1}{x}}{-1} = -1.$$

Therefore

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = \lim_{x\to 1} e^y = \frac{1}{e}.$$

3. (a)

$$\begin{aligned} f(x) &= e^{\cos 2x} \\ f'(x) &= -2\sin(2x)\, e^{\cos 2x} \\ f''(x) &= 4e^{\cos 2x}\sin^2(2x) - 4e^{\cos 2x}\cos 2x \end{aligned}$$

Compute that $f(0) = e$, $f'(0) = 0$, $f''(0) = -4e$, $f'''(0) = 0$. Therefore

$$P_3(x) = e - 2ex^2.$$

(b)

$$\begin{aligned} f(x) &= e^{2x}\ln(1-2x) \\ f'(x) &= 2e^{2x}\ln(1-2x) - \frac{2e^{2x}}{1-2x} \\ f''(x) &= 4e^{2x}\ln(1-2x) + \frac{16xe^{2x}}{(1-2x)^2} - \frac{12e^{2x}}{(1-2x)^2} \\ f'''(x) &= 8e^{2x}\ln(1-2x) - \frac{24e^{2x}}{1-2x} - \frac{24e^{2x}}{(1-2x)^2} - \frac{16e^{2x}}{(1-2x)^3}. \end{aligned}$$

Then $f(0) = 0$, $f'(0) = -2$, $f''(0) = -12$, $f'''(0) = -64$. Then

$$P_3(x) = -2x - 6x^2 - \frac{32x^3}{3}.$$

(c)

$$\begin{aligned} f(x) &= \sec x \\ f'(x) &= \tan x \sec x \\ f''(x) &= \sec^3 x + \tan^2 x \sec x \\ f'''(x) &= 5\tan x \sec^3 x + \tan^3 x \sec x. \end{aligned}$$

Then $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 0$. Therefore

$$P_3(x) = 1 + \frac{x^2}{2}.$$

4. (a) Since for $|y| < 1$,

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots = \sum_{n=0}^{\infty} y^n.$$

Therefore when $|x| < \sqrt{2}$,

$$\frac{4}{2-x^2} = 2\cdot\frac{1}{1-\frac{x^2}{2}} = 2\left(1 + \frac{x^2}{2} + (\frac{x^2}{2})^2 + (\frac{x^2}{2})^3 + \cdots\right) = \sum_{n=0}^{\infty} 2^{1-n}x^{2n}.$$

(b) $f(x) = (1+x)^{\frac{1}{2}}$, then $f'(x) = \dfrac{1}{2}(1+x)^{-\frac{1}{2}}$, $f^{(n)}(x) = (-1)^{n+1}\dfrac{(2n-3)!!}{2^n}(1+x)^{-\frac{2n-1}{2}}$ for $n \geq 2$. In particular, $f(0) = 1$, $f'(0) = \dfrac{1}{2}$, $f^{(n)}(0) = (-1)^{n+1}\dfrac{(2n-3)!!}{2^n}$ for $n \geq 2$. Therefore, when $|x| < 1$,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{(2n-3)!!}{2^n \cdot n!}x^n$$

(c) Since for $|y| < 1$,

$$\ln(1+y) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}y^n.$$

Therefore when $|x| < \frac{4}{3}$,

$$\ln(4+3x) = 2\ln 2 + \ln(1+\frac{3}{4}x) = 2\ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\frac{3}{4})^n}{n}x^n.$$

(d) Since for $|y| < 1$,

$$\frac{1}{1+y} = 1 - y + y^2 - y^3 + \cdots = \sum_{n=0}^{\infty}(-y)^n.$$

Therefore when $|x| < 1$,

$$f(x) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1+\frac{x}{3}}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty}(-x)^n + \sum_{n=0}^{\infty}(-\frac{x}{3})^n\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(1+\frac{1}{3^n}\right)(-1)^n x^n.$$

5. $f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$.

Note that

$$\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + ...$$

, for $-1 < x < 1$, hence

$$\frac{1}{(1-x)^2} = 1 + 2x + ... + nx^{n-1} + ...$$

and

$$\frac{1}{(1-x)^3} = 1 + 3x - ... + \frac{(n+1)(n+2)}{2}x^n + ...$$

6. Since $\sin y = y - \dfrac{y^3}{6} + \dfrac{y^5}{120} + \cdots$, we have (set $y = x^2$)

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{6} + \frac{(x^2)^5}{120} + \cdots = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} + \cdots$$

$$x\sin x = x^2 - x\frac{x^3}{6} + x\frac{x^5}{120} + \cdots = x^2 - \frac{x^4}{6} + \frac{x^6}{120} + \cdots$$

Therefore

$$\sin(x^2) - x\sin x = \frac{x^4}{6} + h.o.t.$$

$$\lim_{x\to 0}\frac{\sin(x^2) - x\sin x}{x^4} = \frac{1}{6}.$$

7. (a) Recall the Taylor series at 0 of $sin(x^2)$ is $\sum \frac{(-1)^k(x^2)^{1+2k}}{(1+2k)!}$ Hence,

$$(1+x^2)sin(x^2) = \sum \frac{(-1)^k(1+x^2)(x^2)^{1+2k}}{(1+2k)!} = \sum \frac{(-1)^k x^{2+4k}}{(1+2k)!} + \frac{(-1)^k x^{4+4k}}{(1+2k)!}$$

(b) By part(a),

$$f^{(100)} = \frac{(-1)^2 4}{49!} \cdot 100! = \frac{100!}{49!}$$

and

$$f^{(101)} = 0$$